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DIMENSION SUBGROUPS AND p-TH POWERS IN p-GROUPS

BY

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ABSTRACT

We prove that if the nilpotence class of a p-group is strictly less than p^k then every product of p^k-th powers can be written as the p-th power of an element. Scoppola and Shalev have proven the same thing for groups of class strictly less than $p^k - p^{k-1}$. They also provide an example which proves that ours is the best possible result. This is a generalization of the well known fact that in groups of class strictly less than p every product of p-powers is again a p-th power. Along the way we prove results of independent interest on dimension subgroups of p-groups.

1. Introduction

In [Ma], Mann suggested studying p-groups with the property that the product $x_1^{p^k} x_2^{p^k} \cdots x_n^{p^k}$ can be written in the form y^p for some y for all n and all $x_1, \ldots, x_n$ in the group. Such groups are sometimes called "power-closed" and we will call them k-power-closed for the duration of this introduction. In that paper, Mann studied 1-power-closed groups. The first class of examples of 1-power-closed p-groups was the regular p-groups and Hall proved this in [Ha]. The next class of p-groups proven to be 1-power-closed was the groups in which the $p-1$-st term of the lower central series is contained in the subgroup generated by the elements of the form y^p. This was proven by Arganbright in [Ar]. Note that this assumption requires p odd as the first term of the lower central series is the whole group.

Lubotzky and Mann introduced powerful p-groups in [LM]. For odd primes they are a special case of the groups considered by Arganbright, groups in which

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the second term of the lower central series (the derived subgroup) is contained in the subgroup generated by the p-powers. For $p = 2$, the condition is that the derived subgroup, the second term of the lower central series, is contained in the subgroup generated by the 4-th powers. Lubotzky and Mann prove that these groups are 1-power-closed.

[W1] contains a proof that certain normal subgroups of powerful p-groups with p odd are also 1-power-closed. This was vastly generalized in [GJ], where González and Jaikin prove that every normal subgroup of one of Arganbright's groups (here called "potent") is 1-power-closed. They also prove that every normal subgroup of a powerful 2-group which is contained in the subgroup generated by the squares is also 1-power-closed.

The latest class of examples was provided in [W2]. A p-group in which the p-th term of the lower central series is contained in the subgroup generated by the elements y^{p^2} is 1-power-closed, generalizing Lubotzky and Mann's result on powerful 2-groups.

In [SS], Scoppola and Shalev prove an important result for general k. Theorem B of that paper implies that if the nilpotence class of a group is less than $(p-1)p^k$ then the group is $(k+1)$-power-closed. They actually prove a slightly stronger result than this. We provide a slight generalization of their theorem in Section 3 and the main result of this paper is a strengthening of their result by allowing groups of class less than p^{k+1}.

Power-closed has also been studied in [Sh] and [RS]. However, these papers are not about finite p-groups and so don't quite fit into our discussion here.

Groups of class strictly less than p are regular. That potent groups are 1-power-closed is related to the fact that regular groups are, as groups of class less than $p-1$ are regular. The result of [SS] in the case $k = 0$ is just this fact about potent groups.

The result of [W2] can be seen as filling in the gap in Arganbright's result by generalizing the fact that groups of class $p-1$ are regular and hence 1-power-closed. The main result of this paper generalizes this by proving that groups of class at most $p^k - 1$ are k-power-closed.

In [SS], the authors note that Corollary 3.6 cannot be strengthened very much. In particular, they give an example of a group of class p^k with elements x and y such that $x^{p^k} y^{p^k}$ is not of the form z^p for any z in the group. Therefore our result is best possible.

2. Notation and tools

We will be primarily concerned with the dimension subgroups of a given p-group. The dimension subgroup series is the fastest descending series of G beginning at $D_1(G) = G$ such that $[D_i(G), D_j(G)] \leq D_{i+j}(G)$ and $D_i(G)^p \leq D_{pi}(G)$. Lazard found a closed form for the dimension subgroups, $D_k(G) = \prod_{ip^j \geq k} \gamma_i(G)^{p^j}$. Let us pause now to review the notation.

For a p-group G and a positive integer j the subgroup G^{p^j} is the subgroup generated by the elements g^{p^j} for $g \in G$. Commonly, this subgroup will contain elements not of the form g^{p^j}. We also recursively define a series of subgroups of G by $\mho^{(0)}(G) = G$ and $\mho^{(\ell+1)}(G) = (\mho^{(\ell)}(G))^p$. It is clear that $G^{p^j} \leq \mho^{(j)}(G)$ but these subgroups need not be equal.

Another important series in the group G is the lower central series. This is the series defined by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$. For subgroups H and K of G we define $[H, K; 0] = H$ and $[H, K; \ell+1] = [[H, K; \ell], K]$. Therefore, $\gamma_{\ell+1}(G) = [G, G; \ell]$.

We will make much use of properties of powerful and potent p-groups. If p is odd then the p-group G is powerful if $\gamma_2(G) \leq G^p$ and potent if $\gamma_{p-1}(G) \leq G^p$. A 2-group is powerful if $\gamma_2(G) \leq G^4$. As in Exercise 2.4.i of [DdMS], this condition is equivalent to the condition that $\gamma_2(G) \leq (G^2)^2$. The following result on potent and powerful groups is very important to us.

PROPOSITION 2.1: *If G is a powerful or potent p-group then $G^{p^k} = \{x^{p^k} : x \in G\}$ for all k. Consequently, $\mho^{(k)}(G) = G^{p^k}$.*

We also require the following result from [GJ].

PROPOSITION 2.2: *If G is a potent p-group for an odd prime p and M and N are normal subgroups of G then $(MN)^p = M^p N^P$.*

A normal subgroup N of G is said to be powerfully embedded in G if $[N, G] \leq N^p$ for odd primes or if $[N, G] \leq N^4$ if $p = 2$. Again, for $p = 2$ it suffices to prove that $[N, G] \leq (N^2)^2$. We say that N is potently embedded if $[N, G; p-2] \leq N^p$.

One consequence of Lazard's closed formula for the dimension subgroups is the following, which is essentially Proposition 11.12 of [DdMS]. We adopt the notation that $n^* = \lceil n/p \rceil$ is the least integer such that $pn^* \geq n$.

LEMMA 2.3: *If G is a finite p-group then $D_n(G) = \gamma_n(G) D_{n^*}(G)^p$.*

Many results on commutators of dimension subgroups are known. Each is essentially a consequence of the following result of P. Hall.

LEMMA 2.4: *For any group G, elements x and y of G, prime p, and positive integer k the following relations hold:*
1. $(xy)^{p^k} \equiv x^{p^k} y^{p^k}$ *modulo* $\gamma_2(\langle x, y\rangle)^{p^k} \cdot \prod_{i=1}^{k} \gamma_{p^i}(\langle x, y\rangle)^{p^{k-i}}$,
2. $[x^{p^k}, y] \equiv [x, y]^{p^k}$ *modulo* $\gamma_2(\langle x, [x, y]\rangle)^{p^k} \cdot \prod_{i=1}^{k} \gamma_{p^i}(\langle x, [x, y]\rangle)^{p^{k-i}}$.

One consequence of this result is that if M and N are normal subgroups of G then $[M^p, N] \leq [M, N]^p[N, M; p]$ and $(MN)^p \leq M^p N^p [M, N, MN; p-2]$.

It is a consequence of the Three Subgroup Lemma that $[M, \gamma_k(N)] \leq [M, N; k]$ for any two normal subgroups M and N of G. The following is Lemma 11.14 of [DdMS].

LEMMA 2.5: *If G is a finite p-group and a and b are positive integers then*
1. $[D_a(G), \gamma_b(G)] \leq [D_a(G), G; b] \leq \gamma_{a+b}(G)\, D_{a+pb}(G)$,
2. *If $a \geq b$ then* $[D_a(G), D_b(G)] \leq \gamma_{a+b}(G)\, D_{a+pb}(G)$.

We will need a somewhat more robust result than this, which we prove in the following lemma. Part (3) of the following is slightly stronger than Lemma 3.2 of [SS]. Note that in our proof we begin our standard practice of writing D_k for $D_k(G)$ and Γ_k for $\gamma_k(G)$.

LEMMA 2.6: *If G is a finite p-group and a, b, and ℓ are positive integers then*
1. $[D_a(G), \gamma_b(G); \ell] \leq \gamma_{a+\ell b}(G)\, D_{a+\ell pb}(G)$,
2. $[\gamma_b(G), D_a(G); \ell] \leq \gamma_{\ell a+b}(G)\, D_{((\ell-1)p+1)a+pb}(G)$,
3. *If $a \geq b$ then* $[D_a(G), D_b(G); \ell] \leq \gamma_{a+\ell b}(G)\, D_{a+\ell pb}(G)$. *In particular,* $\gamma_\ell(D_a(G)) \leq \gamma_{\ell a}(G)\, D_{(p(\ell-1)+1)a}(G)$.

Proof: The Three Subgroup Lemma implies that $[D_a, \Gamma_b; \ell] \leq [D_a, G; \ell b]$ and then (1) follows directly from Lemma 2.5.1.

We prove (2) by induction on ℓ. The case $\ell = 1$ is exactly Lemma 2.5.1. We therefore assume the result holds for ℓ and attempt to prove it for $\ell + 1$. We know that $[\Gamma_b, D_a; \ell + 1] = [[\Gamma_b, D_a; \ell], D_a]$ and hence the induction hypothesis implies that $[\Gamma_b, D_a; \ell + 1] \leq [\Gamma_{\ell a+b} D_{((\ell-1)p+1)a+pb}, D_a]$. This last is contained in the product of $[\Gamma_{\ell a+b}, D_a]$ and $[D_{((\ell-1)p+1)a+pb}, D_a]$.

Lemma 2.5.1 implies that $[\Gamma_{\ell a+b}, D_a]$ is contained in $\Gamma_{(\ell+1)a+b} D_{(p\ell+1)a+pb}$ as desired. Also, Lemma 2.5.2 implies that $[D_{((\ell-1)p+1)a+pb}, D_a]$ is contained in $\Gamma_{((\ell-1)p+2)a+pb} D_{(\ell p+1)a+pb}$. The latter is exactly as desired and the former is contained in $\Gamma_{(\ell+1)a+b}$, completing the proof of (2).

We will also prove (3) by induction on ℓ and here the case $\ell = 1$ is just Lemma 2.5.2. We therefore assume that the result is true for ℓ and prove it for $\ell + 1$. We know that $[D_a, D_b; \ell + 1] = [[D_a, D_b; \ell], D_b]$ and hence the induction hypothesis implies that $[D_a, D_b; \ell+1] \leq [\Gamma_{a+\ell b} D_{a+\ell pb}, D_b]$. This last is contained in the product of $[\Gamma_{a+\ell b}, D_b]$ and $[D_{a+\ell pb}, D_b]$.

Lemma 2.5.1 implies that $[\Gamma_{a+\ell b}, D_b]$ is contained in $\Gamma_{a+(\ell+1)b} D_{pa+(p\ell+1)b}$. The first is exactly as desired and the second is equal to $D_{a+p(\ell+1)b+(p-1)(a-b)}$ and hence contained in $D_{a+p(\ell+1)b}$. Also, Lemma 2.5.2 implies that $[D_{a+\ell pb}, D_b]$ is contained in $\Gamma_{a+(p\ell+1)b} D_{a+p(\ell+1)b}$. The latter is exactly as desired and the former is certainly contained in $\Gamma_{a+(\ell+1)b}$. This completes the proof of (3). ∎

We now wish to prove a lemma that is a more refined version of Lemmas 2.5 and 2.6. To do so, we will need Theorem 2.3(1) of [SS].

THEOREM 2.7: *If G is a finite p-group then*

$$[D_k(G), G; \ell] = \prod_{ip^j \geq k} \gamma_{i+\ell}(G)^{p^j}.$$

Recall that $n^* = \lceil n/p \rceil$ is the least integer at least as big as n/p.

LEMMA 2.8: *Let G be a finite p-group.*

1. $[D_a(G), \gamma_b(G)] \leq \gamma_{a+b}(G)\, \gamma_{a^*+b}(G)^p\, D_{a+p^2b}(G)$ *for any positive integers a and b.*
2. $[D_a(G), \gamma_b(G); \ell] \leq \gamma_{a+\ell b}(G)\, \gamma_{a^*+\ell b}(G)^p\, D_{a+p^2\ell b}(G)$ *for any positive integers a, b, and ℓ.*
3. $[D_a(G), D_b(G)] \leq \gamma_{a+b}(G)\, \gamma_{a^*+b}(G)^p\, D_{a+p^2b}(G) D_{pa+pb}(G)$ *for any positive integers $a \geq b$.*

Proof: We know that $a < p^k$ for some k. As a consequence of the Three Subgroup Lemma,

$$\begin{aligned}[D_a(G), \gamma_b(G)] &\leq [D_a(G), G; b] \\ &= \Gamma_{a+b} \Gamma^p_{a^*+b} \prod_{i=2}^{k} \Gamma^{p^i}_{\lceil a/p^i \rceil + b} \quad \text{(Theorem 2.7)} \\ &\leq \Gamma_{a+b} \Gamma^p_{a^*+b} \prod_{i=2}^{k} D_{a+p^ib} = \Gamma_{a+b} \Gamma^p_{a^*+b} D_{a+p^2b}.\end{aligned}$$

That completes the proof of (1) and provides the base case in an inductive proof of (2). Assume now that the result holds for ℓ. In order to prove the result for $\ell+1$, it suffices to prove that each of $[\Gamma_{a+\ell b}, \Gamma_b]$, $[\Gamma^p_{a^*+\ell b}, \Gamma_b]$, and $[D_{a+p^2\ell b}, \Gamma_b]$ is contained in $\Gamma_{a+(\ell+1)b} \Gamma^p_{a^*+(\ell+1)b} D_{a+p^2(\ell+1)b}$. The first commutator is contained in $\Gamma_{a+(\ell+1)b}$.

By Hall's result,

$$[\Gamma^p_{a^*+\ell b},\Gamma_b]\le[\Gamma_{a^*+\ell b},\Gamma_b]^p[\Gamma_b,\Gamma_{a^*+\ell b};p]$$
$$\le\Gamma^p_{a^*+(\ell+1)b}\Gamma_{b+pa^*+p\ell b}\le\Gamma^p_{a^*+(\ell+1)b}\Gamma_{a+(\ell+1)b}.$$

From (1),

$$[D_{a+p^2\ell b},\Gamma_b]\le\Gamma_{a+(p^2\ell+1)b}\Gamma^p_{a^*+(p\ell+1)b}D_{a+p^2(\ell+1)b}$$
$$\le\Gamma_{a+(\ell+1)b}\Gamma^p_{a^*+(\ell+1)b}D_{a+p^2(\ell+1)b}.$$

Thus we have proven (2) and now wish to prove (3). We will accomplish this by proving that if $ip^j\ge b$ then $[x^{p^j},y]\in\Gamma_{a+b}\Gamma^p_{a^*+b}D_{a+p^2b}D_{pa+pb}$ for all $x\in\Gamma_i$ and $y\in D_a$. We will do this by cases on j.

If $j=0$ then $x\in\Gamma_b$ and the desired result follows from (1). For $j=1$, we have that $ip\ge b$ and $[x^p,y]\equiv[x,y]^p$ modulo $\gamma_2(\langle x,[x,y]\rangle)^p\gamma_p(\langle x,[x,y]\rangle)$. Therefore, $[x^p,y]\in[D_a,\Gamma_i]^p[D_a,\Gamma_i;p]$ and hence it suffices to prove that both of these subgroups are in the desired subgroup.

By (2),

$$[D_a,\Gamma_i;p]\le\Gamma_{a+pi}\Gamma^p_{a^*+pi}D_{a+p^3i}\le\Gamma_{a+b}\Gamma^p_{a^*+b}D_{a+p^2b}.$$

By Lemma 2.5.1, $[D_a,\Gamma_i]\le\Gamma_{a+i}D_{a+pi}\le D_{a+i}$ and hence by Hall's result,

$$[D_a,\Gamma_i]^p\le\Gamma^p_{a+i}D^p_{a+pi}[\Gamma_{a+i},D_{a+pi},D_{a+i};p-2]$$
$$\le\Gamma^p_{a+i}D_{pa+p^2i}[\Gamma_{2a+(p+1)i}D_{(p+1)a+2pi},D_{a+i};p-2]$$

by Lemma 2.5.1.

As $a\ge b$ it follows that $a^*+b\le a+b^*$ and hence $\Gamma^p_{a+i}\le\Gamma^p_{a^*+b}$. Also, $D_{pa+p^2i}\le D_{pa+pb}$. From Lemma 2.6.2,

$$[\Gamma_{2a+(p+1)i},D_{a+i};p-2]\le\Gamma_{pa+(2p-1)i}D_{(p^2-p+1)a+(2p^2-2p+1)i}$$
$$=\Gamma_{a+pi+(p-1)(a+i)}D_{pa+p^2i+(p-1)^2(a+i)}$$
$$\le\Gamma_{a+b}D_{pa+pb}.$$

Finally, Lemma 2.6.3 implies that

$$[D_{(p+1)a+2pi},D_{a+i};p-2]\le\Gamma_{(2p-1)a+(3p-2)i}D_{(p^2-p+1)a+p^2i}$$
$$=\Gamma_{a+pi+2(p-1)(a+i)}D_{pa+p^2i+(p-1)^2a}$$
$$\le\Gamma_{a+b}D_{pa+pb}.$$

This completes the proof for $j=1$.

We now assume $j > 1$ and $ip^j \geq b$. For $x \in \Gamma_i$ and $y \in D_a$, Hall's result implies that $[x^{p^j}, y]$ is in $[\Gamma_i, D_a]^{p^j} \prod_{k=1}^{j}[D_a, \Gamma_i; p^k]^{p^{j-k}}$. As $a \geq b$, we have that $[\Gamma_i, D_a]^{p^j} \leq D_{p^j(a+i)} \leq D_{p^j a+b} \leq D_{a+p^2 b}$. Also, for $k \leq j-2$ we have that $[D_a, \Gamma_i; p^k] \leq D_{a+p^k i}$ and hence $[D_a, \Gamma_i; p^k]^{p^{j-k}} \leq D_{p^{j-k}a+p^j i} \leq D_{p^2 a+b} \leq D_{a+p^2 b}$.

It therefore only remains for us to prove that $[D_a, \Gamma_i; p^{j-1}]^p$ and $[D_a, \Gamma_i; p^j]$ are in the desired subgroup. For the latter, (2) implies that

$$[D_a, \Gamma_i; p^j] \leq \Gamma_{a+p^j i}\Gamma^p_{a^*+p^j i}D_{a+p^{j+2}i} \leq \Gamma_{a+b}\Gamma^p_{a^*+b}D_{a+p^2 b}.$$

Lemma 2.6.1 implies that $[D_a, \Gamma_i; p^{j-1}] \leq \Gamma_{a+p^{j-1}i}D_{a+p^j i}$ and hence Hall's result implies that

$$\begin{aligned}[D_a, \Gamma_i; p^{j-1}]^p &\leq \Gamma^p_{a+p^{j-1}i}D^p_{a+p^j i}[\Gamma_{a+p^{j-1}i}, D_{a+p^j i}, D_{a+p^{j-1}i}; p-2] \\ &\leq \Gamma^p_{a+b^*}D_{pa+p^{j+1}i}[\Gamma_{a+p^{j-1}i}, D_{a+p^j i}, D_{a+p^{j-1}i}; p-2].\end{aligned}$$

As before, $a+b^* \geq a^*+b$ and $pa+p^{j+1}i \geq pa+pb$ so we need only show that $[\Gamma_{a+p^{j-1}i}, D_{a+p^j i}, D_{a+p^{j-1}i}; p-2]$ is in the desired subgroup. Lemma 2.5.1 implies that $[\Gamma_{a+p^{j-1}i}, D_{a+p^j i}] \leq \Gamma_{2a+p^{j-1}(p-1)i}D_{(p+1)a+2p^j i}$ and hence we need only show that $[\Gamma_{2a+p^{j-1}(p-1)i}, D_{a+p^{j-1}i}; p-2]$ and $[D_{(p+1)a+2p^j i}, D_{a+p^{j-1}i}; p-2]$ are contained in the desired subgroup.

Substituting $p = 2$ these two subgroups become $\Gamma_{2a+2^{j-1}i} \leq \Gamma_{2a} \leq \Gamma_{a+b}$ and $D_{3a+2^{j+1}i} \leq D_{2a+2b}$. Hence we may assume $p \geq 3$ for the rest of the proof.

Lemma 2.6.2 implies that

$$\begin{aligned}&[\Gamma_{2a+p^{j-1}(p-1)i}, D_{a+p^{j-1}i}; p-2] \\ &\qquad \leq \Gamma_{pa+p^{j-1}(2p-3)i}D_{(p^2-p+1)a+p^{j-1}(2p^2-4p+1)i} \\ &\qquad = \Gamma_{a+p^j i+(p-1)a+(p-3)p^{j-1}i}D_{pa+p^{j-1}(2p^2-4p+1)i+a(p-1)^2} \\ &\qquad \leq \Gamma_{a+b}D_{pa+p^j(2p-4)i+a(p-1)^2} \\ &\qquad \leq \Gamma_{a+b}D_{pa+(2p-4)b+a(p-1)^2} \\ &\qquad = \Gamma_{a+b}D_{pa+pb+(p-1)^2 a+(p-4)b}.\end{aligned}$$

For $p \geq 5$, the last is clearly contained in D_{pa+pb}. For $p = 3$, this subgroup is $D_{3a+3b+4a-b}$. As $a \geq b$ we know $4a - b \geq 0$ and hence $D_{3a+3b+4a-b} \leq D_{3a+3b}$.

Lemma 2.6.3 implies that

$$\begin{aligned}[D_{(p+1)a+2p^j i}, D_{a+p^{j-1}i}; p-2] &\leq \Gamma_{(2p-1)a+p^{j-1}(3p-2)i}D_{(p^2-p+1)a+p^{j+1}i} \\ &= \Gamma_{a+p^j i+2(p-1)(a+p^{j-1}i)}D_{pa+p^{j+1}i+(p-1)^2 a} \\ &\leq \Gamma_{a+b}D_{pa+pb}\end{aligned}$$

and this completes the proof of (3) and with it the proof of Lemma 2.8. ∎

We will be working with the hypothesis that $\gamma_m(G) \leq D_{p^h}(G)$ for some m and h such that $m < p^h$. Under this hypothesis $D_m(G) = D_{m+1}(G)$ unless m is divisible by p. If m is divisible by p then we can only conclude that $D_{m+1}(G) = D_{m+2}(G)$. We require a version of Theorem 2.5.1 of [SS] which has as its hypothesis that two dimension subgroups are equal. This suffices for our needs except for one particular case, which we prove.

LEMMA 2.9: *If G is a finite p-group such that $\gamma_m(G) \leq D_{p^h}(G)$ with $m < p^h$ then $D_n(G) = D_{n^*}(G)^p$ if $n \geq m+1$. If $\gamma_{p^k}(G) \leq D_{p^{k+1}}(G)$ then $D_{p^k}(G) = D_{p^{k-1}}(G)^p$.*

Proof: Suppose $\Gamma_{p^k} \leq D_{p^{k+1}}$. Lemma 2.3 implies that $\Gamma_{p^k} \leq \Gamma_{p^{k+1}} D^p_{p^k}$ and hence $\Gamma_{p^k} \leq D^p_{p^k}$ due to the following lemma that we will often use without reference.

LEMMA 2.10: *If N is a normal subgroup of a p-group G then $N \leq N^p[N,G]M$ implies that $N \leq M$.*

To complete the proof of Lemma 2.9, note that Lemma 2.3 implies that

$$D_{p^k} \leq \Gamma_{p^k} D^p_{p^k-1} \leq D^p_{p^k} D^p_{p^k-1} = D^p_{p^k-1}. \quad ∎$$

3. Powerful and potent dimension subgroups

Chapter 11 of [DdMS] includes a proof of Lazard's result that if a finitely generated pro-p group G satisfies $\gamma_m(G) \leq G^{p^h}$ with $m < p^h$ then G is p-adic analytic. The technique used there is to conclude that $D_n(G) = D_{n+1}(G)$ for some n (specifically, $n = m$ if p does not divide m and $n = m+1$ otherwise, as in the previous section). Lazard's result is proven by finding that certain subgroups of G are powerful. The following theorem of David Riley is Theorem 11.5 of [DdMS].

THEOREM 3.1: *If G is a finitely generated pro-p group and $D_n(G) = D_{n+1}(G)$ then $D_i(G)$ is powerful for $i \geq n - (n+1)/p$ if p is odd or for $i \geq n$ if $p = 2$.*

We will give a version of this for odd primes under the hypothesis that $\gamma_m(G) \leq D_{p^h}(G)$ with $m < p^h$ and find that many more dimension subgroups are powerful. We will not have use of such a general result on 2-groups, though such a result can be found in [W3], where it is of use in discussing 2-groups of given rank. In Section 5 we will discuss our results in the context of pro-p groups.

THEOREM 3.2: *If p is odd and G is a finite p-group such that $\gamma_m(G) \le D_{p^h}(G)$ with $m < p^h$ then $D_i(G)$ is powerful if $i \ge m/2$ and $i \ge m - p^{h-1}$.*

Proof: For such an i:

$$\begin{aligned}\gamma_2(D_i) &\le \Gamma_{2i} D_{(p+1)i} \quad \text{(Lemma 2.5.2)}\\ &\le \Gamma_{2i}\Gamma_{(p+1)i} D_i^p \quad \text{(Lemma 2.3)}\\ &\le \Gamma_{2i} D_i^p.\end{aligned}$$

It therefore remains for us to prove that $\Gamma_{2i} \le D_i^p$. However, as $2i \ge m$ by assumption,

$$\begin{aligned}\Gamma_{2i} \le [D_{p^h}, G; 2i-m] &= \prod_{j=0}^{h} \Gamma_{p^j+2i-m}^{p^{h-j}} \quad \text{(Theorem 2.7)}\\ &= \Gamma_{2i+p^h-m} \prod_{j=0}^{h-1} \Gamma_{p^j+2i-m}^{p^{h-j}}.\end{aligned}$$

Hence

$$\begin{aligned}\Gamma_{2i} \le \prod_{j=0}^{h-1} \Gamma_{p^j+2i-m}^{p^{h-j}} &\le \prod_{j=0}^{h-1} D_{p^{h-1}+p^{h-j-1}(2i-m)}^p\\ &= D_{p^{h-1}+2i-m}^p = D_{i+i-(m-p^{h-1})}^p \le D_i^p. \quad \blacksquare\end{aligned}$$

A word about the hypothesis that $i \ge m - p^{h-1}$. If $m < p^{h-1}$ then results in this case would normally be handled under the assumption $\gamma_m(G) \le D_{p^{h-1}}$ though, of course, our proof goes through entirely.

A very similar argument allows us to conclude that certain dimension subgroups are potent.

THEOREM 3.3: *If p is odd and G is a finite p-group such that $\gamma_m(G) \le D_{p^h}(G)$ with $m < p^h$ then $D_i(G)$ is potent if $i \ge m/(p-1)$ and $i \ge (m - p^{h-1})/(p-2)$.*

Proof: For such an i:

$$\begin{aligned}\gamma_{p-1}(D_i) &\le \Gamma_{(p-1)i} D_{(p-1)^2 i} \quad \text{(Lemma 2.6.3)}\\ &\le \Gamma_{(p-1)i}\Gamma_{(p-1)^2 i} D_{(p-2)i}^p \quad \text{(Lemma 2.3)}\\ &\le \Gamma_{(p-1)i} D_i^p\end{aligned}$$

where we use that p is odd.

It therefore remains for us to prove that $\Gamma_{(p-1)i} \leq D_i^p$. However,

$$\Gamma_{(p-1)i} \leq [D_{p^h}, G; (p-1)i - m] = \prod_{j=0}^{h} \Gamma^{p^{h-j}}_{p^j+(p-1)i-m} \quad \text{(Theorem 2.7)}$$
$$= \Gamma_{(p-1)i+p^h-m} \prod_{j=0}^{h-1} \Gamma^{p^{h-j}}_{p^j+(p-1)i-m}.$$

Hence,

$$\Gamma_{(p-1)i} \leq \prod_{j=0}^{h-1} \Gamma^{p^{h-j}}_{p^j+(p-1)i-m} \leq \prod_{j=0}^{h-1} D^p_{p^{h-1}+p^{h-j-1}((p-1)i-m)}$$
$$= D^p_{p^{h-1}+(p-1)i-m} = D^p_{i+(p-2)i-(m-p^{h-1})} \leq D_i^p. \quad ∎$$

The previous two arguments can be generalized to give

PROPOSITION 3.4: *If p is odd and G is a finite p-group such that $\gamma_m(G) \leq D_{p^h}(G)$ with $m < p^h$ then $\gamma_k(D_i(G)) \leq D_i(G)^p$ if $i \geq m/k$ and $i \geq (m - p^{h-1})/(k-1)$.*

The proof of the following proposition is essentially the same as the proof of Theorem B of [SS], though stated more generally. We will give (a slight generalization of the statement of) Theorem B as a corollary. Recall that we define $\mho^{(0)}(G) = G$ and $\mho^{(i+1)}(G) = (\mho^{(i)}(G))^p$. A quite simple induction gives that $\mho^{(i)}(G) \leq D_{p^i}(G)$.

PROPOSITION 3.5: *Let G be a finite p-group with p odd. If there exists k such that $\gamma_{(p-1)p^k}(G) \leq D_{p^{k+1}}(G)$ then $D_{p^{k+\ell}}(G) \subseteq \{x^{p^\ell} : x \in G\}$ for positive integers ℓ.*

Proof: Under the assumption, Theorem 3.3 implies that D_{p^k} is potent. Repeated applications of Lemma 2.9 imply that $D_{p^{k+\ell}} = \mho^{(\ell)}(D_{p^k})$ and Proposition 2.1 implies that $\mho^{(\ell)}(D_{p^k}) = \{x^{p^\ell} : x \in D_{p^k}\}$. This completes the proof. ∎

COROLLARY 3.6: *Let G be a finite p-group with p odd with nilpotence class c. Let k be the minimal integer such that $c < (p-1)p^k$. Then $D_{p^{k+\ell}}(G) \subseteq \{x^{p^\ell} : x \in G\}$ for positive integers ℓ.*

4. Proof of the main result

We will prove the following theorem which is stronger than Proposition 3.5. It also handles the case of $p = 2$ which is not addressed in that proposition.

THEOREM 4.1: *Let G be a finite p-group such that $\gamma_{p^k}(G) \leq D_{p^{k+1}}(G)$ for some k. Then $D_{p^{k+\ell-1}}(G) \subseteq \{x^{p^\ell} : x \in G\}$ for positive integers ℓ.*

COROLLARY 4.2: *Let G be a finite p-group with nilpotence class c. Let k be the minimal integer such that $c < p^{k+1}$. Then $D_{p^{k+\ell}}(G) \subseteq \{x^{p^\ell} : x \in G\}$ for positive integers ℓ.*

We will prove this theorem first for odd primes and then when $p = 2$. The first instance of this, when $k = 1$, is a consequence of the main result of [W2].

THEOREM 4.3: *Suppose that G is a finite p-group such that $\gamma_p(G) \leq G^{p^2}$. Then $G^p = \{x^p : x \in G\}$.*

Let us now prove that Theorem 4.1 holds with $k = 1$.

PROPOSITION 4.4: *Let G be a finite p-group such that $\gamma_p(G) \leq D_{p^2}(G)$. Then $D_{p^\ell}(G) \subseteq \{x^{p^\ell} : x \in G\}$ for positive integers ℓ.*

Proof: We first note that $D_{p^2} = \Gamma_{p^2}\Gamma_p^p G^{p^2}$. Our assumption that $\Gamma_p \leq D_{p^2}$ is therefore equivalent to the assumption that $\Gamma_p \leq G^{p^2}$. Also, $D_p = \Gamma_p G^p$ and so our assumption implies that $D_p = G^p = \{x^p : x \in G\}$. If p is odd then Theorem 3.2 implies that D_p is powerful. If $p = 2$ then G is powerful and standard results imply that G^2 is also powerful. Repeated applications of Lemma 2.9 imply that $D_{p^\ell} = \mho^{(\ell-1)}(D_p)$ and hence every element of D_{p^ℓ} is a $p^{\ell-1}$-power of an element of D_p and hence a p^ℓ-power of an element of G. ∎

We will now work with $k \geq 2$. For the sake of convenience, however, we will replace k by $k - 1$.

PROPOSITION 4.5: *Let G be a finite p-group with p odd. If $\gamma_{p^{k+1}}(G) \leq D_{p^{k+2}}(G)$ for some k with $k \geq 1$ then $D_{p^{k+\ell}}(G) \subseteq \{x^{p^\ell} : x \in G\}$ for positive integers ℓ.*

Proof: We first note that Theorems 3.2 and 3.3 imply that D_i is powerful for $i \geq p^{k+1}/2$ and D_i is potent for $i \geq p^{k+1}/(p-1)$. In particular, $D_{p^{k+1}}$ is powerful and D_{2p^k} is potent. Also, Lemma 2.9 implies that $D_{p^{k+1}} = D_{p^k}^p$ and $D_{p^{k+2}} = D_{p^{k+1}}^p$. In particular, $\Gamma_{p^{k+1}} \leq D_{p^{k+1}}^p$.

Consider $x, y \in D_{p^k}$. Certainly $x^p y^p \equiv (xy)^p$ modulo $\gamma_2(\langle x, y\rangle)^p \gamma_p(\langle x, y\rangle)$ by Hall's result. We find that $\gamma_2(\langle x, y\rangle) \leq [D_{p^k}, D_{p^k}] \leq D_{2p^k}$ which is potent.

Lemma 2.5.2 implies that $[D_{p^k}, D_{p^k}] \leq \Gamma_{2p^k} D_{(p+1)p^k}$ and so Proposition 2.2 implies that $[D_{p^k}, D_{p^k}]^p \leq \Gamma^p_{2p^k} D^p_{(p+1)p^k}$.

Also, $\gamma_p(\langle x, y\rangle) \leq \gamma_p\left(D_{p^k}\right)$. Now $\gamma_{p-1}\left(D_{p^k}\right) \leq \Gamma_{(p-1)p^k} D_{(p-1)^2 p^k}$ by Lemma 2.6.3. Therefore, $\gamma_p\left(D_{p^k}\right) \leq [\Gamma_{(p-1)p^k} D_{(p-1)^2p^k}, D_{p^k}]$. This latter subgroup is contained in the product of $[\Gamma_{(p-1)p^k}, D_{p^k}]$ and $[D_{(p-1)^2p^k}, D_{p^k}]$. We investigate these two subgroups in turn.

Lemma 2.8.1 implies that

$$\begin{aligned}[]
[\Gamma_{(p-1)p^k}, D_{p^k}] &\leq \Gamma_{p^{k+1}} \Gamma^p_{p^{k-1}(p^2-p+1)} D_{p^k(p^3-p^2+1)} \\
&= \Gamma_{p^{k+1}} \Gamma^p_{2p^k + p^{k-1}(p(p-3)+1)} D_{p^{k+2}+p^k(p^2(p-2)+1)} \\
&\leq \Gamma_{p^{k+1}} \Gamma^p_{2p^k} D_{p^{k+2}} \leq D^p_{p^{k+1}} \Gamma^p_{2p^k} D^p_{p^{k+1}}.
\end{aligned}$$

Lemma 2.8.3 implies that

$$\begin{aligned}
&[D_{p^{k+1}(p-2)}, D_{p^k}] \\
&\leq \Gamma_{p^k(p^2-2p+1)} \Gamma^p_{p^k(p-1)} D_{p^{k+1}(2p-2)} D_{p^{k+1}(p^2-2p+1)} \\
&= \Gamma_{p^{k+1}+p^k(p(p-3)+1)} \Gamma^p_{2p^k+p^k(p-3)} D_{p^{k+2}+p^{k+1}(p-2)} D_{p^{k+2}+p^{k+1}(p(p-3)+1)} \\
&\leq \Gamma_{p^{k+1}} \Gamma^p_{2p^k} D_{p^{k+2}} \leq D^p_{p^{k+1}} \Gamma^p_{2p^k} D^p_{p^{k+1}}.
\end{aligned}$$

As $(p-1)^2 > p-2$ for all primes, this implies that $[D_{(p-1)^2p^k}, D_{p^k}]$ is contained in $\Gamma^p_{2p^k} D^p_{p^{k+1}}$.

Therefore, $x^p y^p \in (\langle xy\rangle \Gamma_{2p^k} D_{p^{k+1}})^p$. We will now prove that $\langle xy\rangle\Gamma_{2p^k} D_{p^{k+1}}$ is potent. Any non-trivial commutator in the generators must have one term from $\Gamma_{2p^k} D_{p^{k+1}}$. Therefore, $\gamma_{p-1}\left(\langle xy\rangle \Gamma_{2p^k} D_{p^{k+1}}\right) \leq [\Gamma_{2p^k} D_{p^{k+1}}, D_{p^k}; p-2]$. We will show that this is contained in $\Gamma^p_{2p^k} D^p_{p^{k+1}}$ proving that the subgroup in question is potent. In fact, we will have proven that $\Gamma_{2p^k} D_{p^{k+1}}$ is potently embedded in D_{p^k}.

Now, $[\Gamma_{2p^k} D_{p^{k+1}}, D_{p^k}; p-2] \leq [\Gamma_{2p^k}, D_{p^k}; p-2][D_{p^{k+1}}, D_{p^k}; p-2]$ as all of our subgroups are normal. We consider these two terms in turn.

Lemma 2.6.2 implies that $[\Gamma_{2p^k}, D_{p^k}; p-3] \leq \Gamma_{(p-1)p^k} D_{p^k(p-1)^2}$. (Notice that this holds for $p = 3$ as well.) Hence, $[\Gamma_{2p^k}, D_{p^k}; p-2]$ is contained in the product of $[\Gamma_{(p-1)p^k}, D_{p^k}]$ and $[D_{p^k(p-1)^2}, D_{p^k}]$. We have already seen that each of these is contained in $\Gamma^p_{2p^k} D^p_{p^{k+1}}$. Therefore $[\Gamma_{2p^k}, D_{p^k}; p-2] \leq \Gamma^p_{2p^k} D^p_{p^{k+1}}$.

We now will prove that $[D_{p^{k+1}}, D_{p^k}; p-2]$ is contained in $\Gamma^p_{2p^k} D^p_{p^{k+1}}$. Lemma 2.6.3 implies that $[D_{p^{k+1}}, D_{p^k}; p-3]$ is contained in $\Gamma_{p^k(2p-3)} D_{p^{k+1}(p-2)}$. (Notice that this continues to hold if $p = 3$.) Hence $[D_{p^{k+1}}, D_{p^k}; p-2]$ is contained in the product of $[\Gamma_{p^k(2p-3)}, D_{p^k}]$ and $[D_{p^{k+1}(p-2)}, D_{p^k}]$.

We have already proven that $[D_{(p-2)p^{k+1}}, D_{p^k}] \leq \Gamma^p_{2p^k} D^p_{p^{k+1}}$. Notice that

$\Gamma_{(2p-3)p^k} = \Gamma_{(p-1+p-2)p^k} \le \Gamma_{(p-1)p^k}$ and therefore we already have proven that $[\Gamma_{(2p-3)p^k}, D_{p^k}] \le \Gamma^p_{2p^k} D^p_{p^{k+1}}$.

We have now proven that $x^p y^p$ is contained in H^p for a potent subgroup H of D_{p^k}. Therefore, $x^p y^p$ is a p-th power of an element of D_{p^k} and we conclude that every element of $D^p_{p^k}$ is a p-th power of an element of D_{p^k}. As $D_{p^{k+1}} = D^p_{p^k}$, every element of $D_{p^{k+1}}$ is a p-th power, giving the result of the proposition for $\ell = 1$.

Repeated applications of Lemma 2.9 imply that $D_{p^{k+\ell}} = \mho^{(\ell-1)}(D_{p^{k+1}})$. As $D_{p^{k+1}}$ is powerful, every element of $\mho^{(\ell-1)}(D_{p^{k+1}})$ is a $p^{\ell-1}$ power of an element of $D_{p^{k+1}}$, and hence a p^ℓ power of an element of D_{p^k}. ∎

The proof of Theorem 4.1 will be complete when we have proven it for the prime 2. This case is divided into two propositions as we will need to use a different technique which requires consideration of 2^{k-2} and we wish to avoid trouble with fractions by considering $k = 1$ separately. Notice that we also prove that certain dimension subgroups are powerful. This cannot be deduced from Theorem 3.1.

Proposition 4.6: *Let G be a finite 2-group such that $\gamma_4(G) \le D_8(G)$. Then $D_3(G)$ is powerfully embedded in $D_2(G)$ and $D_4(G)$ is powerfully embedded in $D_3(G)$. Also, $D_{2^{\ell+1}}(G) \subseteq \{x^{2^\ell} : x \in G\}$ for positive integers ℓ.*

Proof: Note that Lemma 2.9 implies that $D_4 = D_2^2$ and $D_8 = D_4^2$. In particular, $\Gamma_4 \le D_4^2$.

Lemma 2.8.3 implies that $[D_3, D_2] \le \Gamma_5 \Gamma_4^2 D_{10}$. We will prove that $[D_3, D_2]$ is contained in $(D_3^2)^2$ and this suffices to prove that $[D_3, D_2] \le D_3^4$ as in Exercise 2.4.i of [DdMS].

Lemma 2.3 implies that $D_{10} \le \Gamma_{10} D_5^2$ and hence $[D_3, D_2] \le \Gamma_5 \Gamma_4^2 D_5^2$. The lemma also implies that $D_5 \le \Gamma_5 D_3^2$. Thus $D_5^2 \le \Gamma_5^2 (D_3^2)^2 [\Gamma_5, D_3^2] \le \Gamma_5 (D_3^2)^2$. We have thus far proven that $[D_3, D_2] \le \Gamma_5 \Gamma_4^2 (D_3^2)^2$.

Now $\Gamma_5 \le [D_8, G] = \Gamma_9 \Gamma_5^2 \Gamma_3^4 \Gamma_2^8$ by Theorem 2.7 and hence $\Gamma_5 \le \Gamma_3^4 \Gamma_2^8$. Certainly $\Gamma_3^4 \le (D_3^2)^2$ and also $\Gamma_2^8 \le (\Gamma_2^2)^4 \le D_4^4$. We can therefore conclude that $[D_3, D_2] \le \Gamma_4^2 (D_3^2)^2$. As $\Gamma_4 \le D_4^2$, we have that $[D_3, D_2] \le (D_3^2)^2$. Therefore, D_3 is powerfully embedded in D_2 and also powerful.

Lemma 2.8.3 implies that $[D_4, D_3] \le \Gamma_7 \Gamma_5^2 D_{14}$. Also $D_{14} \le \Gamma_{14} D_7^2$ by Lemma 2.3. Hence $[D_4, D_3] \le \Gamma_7 \Gamma_5^2 D_7^2$. The lemma also implies that $D_7 \le \Gamma_7 D_4^2$. Therefore, $D_7^2 \le \Gamma_7^2 (D_4^2)^2 [\Gamma_7, D_4^2] \le \Gamma_7 (D_4^2)^2$ and thus $[D_4, D_3] \le \Gamma_7 \Gamma_5^2 (D_4^2)^2$.

Now $\Gamma_7 \le [D_8, G; 3] = \Gamma_{11} \Gamma_7^2 \Gamma_5^4 \Gamma_4^8$ by Theorem 2.7 and hence $\Gamma_7 \le \Gamma_5^4 \Gamma_4^8 \le D_4^4$. Therefore, $[D_4, D_3] \le \Gamma_5^2 (D_4^2)^2$. As $\Gamma_4 \le D_4^2$, we have that $[D_4, D_3] \le (D_4^2)^2$.

Therefore, D_4 is powerfully embedded in D_3 and also powerful.

Now consider $x, y \in D_2$. Certainly $(xy)^2 = x^2y^2[y,x]^y$. Lemma 2.8.3 implies that

$$[D_2, D_2] \leq \Gamma_4\Gamma_3^2 D_8 \leq D_4^2 D_3^2 D_4^2.$$

Therefore, $x^2y^2 \in (\langle xy\rangle D_3)^2$. As $[D_3, D_2] \leq (D_3^2)^2$, we know that $\langle xy\rangle D_3$ is powerful. Therefore, x^2y^2 is the square of an element of D_2 and we conclude that all elements of $D_4 = D_2^2$ are squares of elements of D_2. As D_4 is powerful by the above, every element of $\mho^{(\ell-1)}(D_4)$ is a $2^{\ell-1}$ power of an element of D_4 and hence a 2^ℓ power of an element of D_2. This completes the proof as $D_{2^{\ell+1}} = \mho^{(\ell-1)}(D_4)$ by Lemma 2.9. ∎

We finish the proof of Theorem 4.1 by handling the general case with $p = 2$.

PROPOSITION 4.7: *Let G be a finite 2-group such that $\gamma_{2^{k+1}}(G) \leq D_{2^{k+2}}(G)$ for some $k \geq 2$. Then $D_{2^{k+1}}(G)$ is powerful and $D_{2^{k+\ell}}(G) \subseteq \{x^{2^\ell} : x \in G\}$.*

Proof: Note that Lemma 2.9 implies that $D_{2^{k+1}} = D_{2^k}^2$ and $D_{2^{k+2}} = D_{2^{k+1}}^2$. In particular, $\Gamma_{2^{k+1}} \leq D_{2^{k+1}}^2$.

We will first prove that $D_{2^{k+1}}$ is powerful. Lemma 2.8.3 implies that

$$\begin{aligned}[D_{2^{k+1}}, D_{2^{k+1}}] \leq \Gamma_{2^{k+2}}\Gamma^2_{2^{k+1}+2^k} D_{2^{k+3}} &= \Gamma_{2^{k+2}}\Gamma^2_{2^{k+1}+2^k} D^2_{2^{k+2}} \quad \text{(Lemma 2.9)}\\ &= \Gamma_{2^{k+2}}\Gamma^2_{2^{k+1}+2^k}(D^2_{2^{k+1}})^2.\end{aligned}$$

We know that $\Gamma_{2^{k+1}+2^{k-1}} \leq [D_{2^{k+2}}, G; 2^{k-1}]$ and hence Theorem 2.7 implies that $\Gamma_{2^{k+1}+2^{k-1}} \leq \Gamma_{2^{k+2}+2^{k-1}}\Gamma^2_{2^{k+1}+2^{k-1}}\Gamma^4_{2^k+2^{k-1}}\prod_{i=3}^{k+2}\Gamma^{2^i}_{2^{k+2-i}+2^{k-1}}$. Therefore, $\Gamma_{2^{k+1}+2^{k-1}}$ is contained in $\Gamma^4_{2^k+2^{k-1}}\prod_{i=3}^{k+2} D^4_{2^k+2^{k+i-3}}$. This last product is equal to the largest subgroup, which occurs when $i = 3$ and hence is $D^4_{2^k+2^k} = D^4_{2^{k+1}}$.

Hence

$$\begin{aligned}\Gamma^2_{2^{k+1}+2^k} \leq (\Gamma^4_{2^k+2^{k-1}}D^4_{2^{k+1}})^2 &\leq (\Gamma^4_{2^k+2^{k-1}})^2(D^4_{2^{k+1}})^2[D^4_{2^{k+1}}, \Gamma^4_{2^k+2^{k-1}}]\\ &\leq (\Gamma^4_{2^k+2^{k-1}})^2 D^4_{2^{k+1}}\\ &\leq (D^2_{2^{k+1}+2^k})^2 D^4_{2^{k+1}} \leq (D^2_{2^{k+1}})^2.\end{aligned}$$

Hence $[D_{2^{k+1}}, D_{2^{k+1}}] \leq \Gamma_{2^{k+2}}(D^2_{2^{k+1}})^2$.

We know that $\Gamma_{2^{k+2}} \leq [D_{2^{k+2}}, G; 2^{k+1}]$ and hence Theorem 2.7 implies that $\Gamma_{2^{k+2}}$ is contained in $\Gamma_{2^{k+2}+2^{k+1}}\Gamma^2_{2^{k+2}}\prod_{i=2}^{k+2}\Gamma^{2^i}_{2^{k+2-i}+2^{k+1}}$. Therefore, $\Gamma_{2^{k+2}}$ is contained in $\prod_{i=2}^{k+2} D^4_{2^k+2^{k+i-1}}$. This product is equal to the largest subgroup, which occurs when $i = 2$ and hence is $D^4_{2^{k+1}+2^k} \leq D^4_{2^{k+1}}$. We can therefore now conclude that $[D_{2^{k+1}}, D_{2^{k+1}}] \leq (D^2_{2^{k+1}})^2$ and hence $D_{2^{k+1}}$ is powerful.

Now consider $x, y \in D_{2^k}$. We know that $(xy)^2 = x^2y^2[y,x]^y$. The commutator $[y,x]$ is in $[D_{2^k}, D_{2^k}]$ and so we consider this subgroup. Lemma 2.8.3 implies that

$$\begin{aligned}[D_{2^k}, D_{2^k}] \leq \Gamma_{2^{k+1}}\Gamma^2_{2^k+2^{k-1}}D_{2^{k+2}} &= \Gamma_{2^{k+1}}\Gamma^2_{2^k+2^{k-1}}D^2_{2^{k+1}} \\ &\leq D^2_{2^{k+1}}\Gamma^2_{2^k+2^{k-1}}D^2_{2^{k+1}}.\end{aligned}$$

Hence $x^2y^2 \in (\langle xy\rangle\Gamma_{2^k+2^{k-1}}D_{2^{k+1}})^2$. We will show that $\langle xy\rangle\Gamma_{2^k+2^{k-1}}D_{2^{k+1}}$ is powerful and hence conclude that x^2y^2 is the square of an element of D_{2^k}. We will do this by proving that $[\Gamma_{2^k+2^{k-1}}, D_{2^k}]$ and $[D_{2^{k+1}}, D_{2^k}]$ are contained in $((\Gamma_{2^k+2^{k-1}}D_{2^{k+1}})^2)^2$.

It is a consequence of the Three Subgroup Lemma that

$$\begin{aligned}[D_{2^k},&\Gamma_{2^k+2^{k-1}}] \\ &\leq [D_{2^k}, G; 2^k + 2^{k-1}] \\ &= \Gamma_{2^{k+1}+2^{k-1}}\Gamma^2_{2^{k+1}}\Gamma^4_{2^k+2^{k-1}+2^{k-2}}\prod_{i=3}^{k}\Gamma^{2^i}_{2^k+2^{k-1}+2^{k-i}} \qquad \text{(Theorem 2.7)} \\ &\leq \Gamma_{2^{k+1}+2^{k-1}}(D^2_{2^{k+1}})^2\Gamma^4_{2^k+2^{k-1}}\prod_{i=3}^{k}D^4_{2^{k+i-2}+2^{k+i-3}+2^{k-2}} \\ &= \Gamma_{2^{k+1}+2^{k-1}}(D^2_{2^{k+1}})^2\Gamma^4_{2^k+2^{k-1}}D^4_{2^{k+1}+2^k+2^{k-2}} \\ &= \Gamma_{2^{k+1}+2^{k-1}}(D^2_{2^{k+1}})^2\Gamma^4_{2^k+2^{k-1}}.\end{aligned}$$

We proved that $\Gamma_{2^{k+1}+2^{k-1}} \leq \Gamma^4_{2^k+2^{k-1}}D^4_{2^{k+1}}$ in our proof that $D_{2^{k+1}}$ is powerful. Therefore, $[\Gamma_{2^k+2^{k-1}}, D_{2^k}] \leq ((\Gamma_{2^k+2^{k-1}}D_{2^{k+1}})^2)^2$.

Also,

$$\begin{aligned}[D_{2^{k+1}}, D_{2^k}] &= [D^2_{2^k}, D_{2^k}] \\ &\leq [D_{2^k}, D_{2^k}]^2[D_{2^k}, D_{2^k}, D_{2^k}] \\ &\leq [D_{2^k}, D_{2^k}]^2[\Gamma^2_{2^k+2^{k-1}}D^2_{2^{k+1}}, D_{2^k}] \quad \text{(as before)} \\ &\leq [D_{2^k}, D_{2^k}]^2[\Gamma^2_{2^k+2^{k-1}}, D_{2^k}][D^2_{2^{k+1}}, D_{2^k}] \\ &\leq [D_{2^k}, D_{2^k}]^2[\Gamma^2_{2^k+2^{k-1}}, D_{2^k}][D_{2^{k+1}}, D_{2^k}]^2[D_{2^{k+1}}, D_{2^k}, D_{2^{k+1}}].\end{aligned}$$

Therefore, $[D_{2^{k+1}}, D_{2^k}]$ is contained in $[D_{2^k}, D_{2^k}]^2[\Gamma^2_{2^k+2^{k-1}}, D_{2^k}]$. We have already seen that $[\Gamma_{2^k+2^{k-1}}, D_{2^k}]$ is contained in $((\Gamma_{2^k+2^{k-1}}D_{2^{k+1}})^2)^2$ and we know that $[D_{2^k}, D_{2^k}]$ is contained in $\Gamma^2_{2^k+2^{k-1}}D^2_{2^{k+1}}$ and hence $[D_{2^k}, D_{2^k}]^2$ is contained in $((\Gamma_{2^k+2^{k-1}}D_{2^{k+1}})^2)^2$.

We have now proven that x^2y^2 is contained in H^2 for a powerful subgroup H of D_{2^k}. Therefore, x^2y^2 is the square of an element of D_{2^k} and we conclude that every element of $D_{2^k}^2$ is the square of an element of D_{2^k}. Hence every element of $D_{2^{k+1}}$ is a square, giving the result of the proposition for $\ell = 1$.

Repeated applications of Lemma 2.9 imply that $D_{2^{k+\ell}} = \mho^{(\ell-1)}(D_{2^{k+1}})$. As $D_{2^{k+1}}$ is powerful, every element of $\mho^{(\ell-1)}(D_{2^{k+1}})$ is a $2^{\ell-1}$ power of an element of $D_{2^{k+1}}$, and hence a 2^ℓ power of an element of D_{2^k}. ∎

5. Pro-p groups

In the introduction, we mentioned Shalev's paper [Sh] which is concerned with pro-p groups. We would like to discuss our results in this context.

Our results all hold in finitely generated pro-p groups. None of our proofs relies on finite nilpotence class so no changes need to be made there. The only remaining concern in translating the proofs is that care occasionally needs to be taken as certain results only hold up to closures of subgroups. However, standard results (see [DdMS]) imply that dimension subgroups and terms of the lower central series and their products are closed. The only other subgroups we consider are of the form $\langle x \rangle N$ where N is an open normal subgroup of G. Therefore, N has finite index and so $\langle x \rangle N$ is a finite union of closed sets and therefore closed as well.

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